Quantum and classical dynamics of Langmuir wave packets

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The quantum Zakharov system in three spatial dimensions and an associated Lagrangian description, as well as its basic conservation laws, are derived. In the adiabatic and semiclassical cases, the quantum Zakharov system reduces to a quantum modified vector nonlinear Schrödinger (NLS) equation for the envelope electric field. The Lagrangian structure for the resulting vector NLS equation is used to investigate the time dependence of the Gaussian-shaped localized solutions, via the Rayleigh-Ritz variational method. The formal classical limit is considered in detail. The quantum corrections are shown to prevent the collapse of localized Langmuir envelope fields, in both two and three spatial dimensions. Moreover, the quantum terms can produce an oscillatory behavior of the width of the approximate Gaussian solutions. The variational method is shown to preserve the essential conservation laws of the quantum modified vector NLS equation. The possibility of laboratory tests in the next generation intense laser-solid plasma compression experiment is discussed.

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I. INTRODUCTION

The Zakharov system [1], describing the coupling between Langmuir and ion-acoustic waves, is one of the basic models in plasma physics (see Refs. [2,3] for reviews). Recently [4], a quantum modified Zakharov system was derived, by means of the quantum plasma hydrodynamic equations [5–7]. Enhancement of the quantum effects was then shown, e.g., to suppress the four-wave decay instability. Subsequently [8], a kinetic treatment of the quantum Zakharov system has proved that the modulational instability growth rate can be increased in comparison to the classical case, for partially coherent Langmuir wave electric fields. Also, a variational formalism was identified and used to study the radiation of localized structures described by the quantum Zakharov system [9]. Bell-shaped electric field envelopes of electron plasma oscillations in dense quantum plasmas obeying Fermi statistics were analyzed in Ref. [10]. More mathematically oriented works on the quantum Zakharov equations concern its Lie symmetry group [11] and the derivation of exact solutions [12–14]. Finally, using Galerkin approximation methods, the appearance of hyperchaos in the temporal dynamics of the quantum Zakharov system was highlighted [15].

All these papers refer to quantum Zakharov equations in one spatial dimension only. In the present work, we extend the quantum Zakharov system to fully three-dimensional (3D) space, allowing also for a magnetic field perturbation. In the classical case, both heuristic arguments and numerical simulations indicate that the ponderomotive force can produce finite-time collapse of Langmuir wave packets in two or three dimensions [2,16,17]. This is in contrast to the one-dimensional case, whose solutions are smooth. A dynamic rescaling method was used for the time evolution of electrostatic self-similar and asymptotically self-similar solutions in two and three dimensions, respectively [18]. Allowing for transverse fields shows that singular solutions of the resulting vector Zakharov equations are weakly anisotropic, for a large class of initial conditions [19]. The electrostatic nonlinear collapse of Langmuir wave packets in the ionospheric and laboratory plasmas has been observed [20,21]. Also, the collapse of Langmuir wave packets in beam plasma experiments verifies the basic concepts of strong Langmuir turbulence, as introduced by Zakharov [22]. The analysis of the coupled longitudinal and transverse modes in the classical strong Langmuir turbulence has been less studied [23–25], as well as the intrinsically magnetized case [26], which can lead to upper-hybrid wave collapse [27]. Finally, Zakharov-like equations have been proposed for the electromagnetic wave collapse in a radiation background [28].

It is expected that the ponderomotive force causing the collapse of localized solutions in two- or three-space dimensions could be weakened by the inclusion of quantum effects, making the dynamics less violent. This conjecture is checked after establishing the quantum Zakharov system in higher-dimensional space and using its variational structure in association with a (Rayleigh-Ritz) trial function method.

The paper is organized in the following fashion. In Sec. II, the quantum Zakharov system in three spatial dimensions is derived by means of the usual two-time-scale method applied to the fully 3D quantum hydrodynamic model. In Sec. III, the 3D quantum Zakharov system is shown to be described by a Lagrangian formalism. The basic conservation laws are then also derived. When the density fluctuations are slow enough so that an adiabatic approximation is possible, and treating the quantum term of the low-frequency equation as a perturbation, a quantum modified vector nonlinear Schrödinger (NLS) equation for the envelope electric field is obtained. In Sec. IV, the variational structure is used to analyze the temporal dynamics of localized (Gaussian) solutions of this quantum NLS equation, through the Rayleigh-Ritz

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method, in two spatial dimensions. Some numerical estimates and simulations concerning the next generation intense laser-solid plasma compression experiment are devised. Section V extended the treatment to fully 3D space. Special attention is paid to the comparison between the classical and quantum cases and the considerable qualitative and quantitative differences between them. Section VI shows the conclusions.

II. QUANTUM ZAKHAROV EQUATIONS IN 3+1 DIMENSIONS

The starting point for the derivation of the electromagnetic quantum Zakharov equations is the quantum hydrodynamic model for an electron-ion plasma, Eqs. (20)–(28) of Ref. [7]. For the electron fluid pressure p_e, consider the equation of state for spin 1/2 particles at zero temperature,

$$p_e = \frac{3}{5}\frac{m_e v_{Fe}^2 n_e^{5/3}}{n_0^{2/3}}, \tag{1}$$

where m_e is the electron mass, v_{Fe} is the electron Fermi speed, n_e is the electron number density, and n_0 is the equilibrium particle number density both for electron and ions. The pressure and quantum effects (due to their larger mass) are neglected for the ions. Also due to the larger ion mass, it is possible to introduce a two-time-scale decomposition, $n_e = n_0 + \delta n_s + \delta n_f$, $n_i = n_0 + \delta n_s$, $\mathbf{u}_e = \delta\mathbf{u}_s + \delta\mathbf{u}_f$, $\mathbf{u}_i = \delta\mathbf{u}_s$, $\mathbf{E} = \delta\mathbf{E}_s + \delta\mathbf{E}_f$, and $\mathbf{B} = \delta\mathbf{B}_f$, where the subscripts s and f refer to slowly and rapidly changing quantities, respectively. Also, $\mathbf{u}_e$ is the electron fluid velocity, n_i is the ion number density, $\mathbf{u}_i$ is the ion fluid velocity, $\mathbf{E}$ is the electric field, and $\mathbf{B}$ is the magnetic field. Notice that it is assumed that there is no slow contribution to the magnetic field, a restriction which allows getting $\mathbf{B} = (m_e/e)\nabla \times \delta\mathbf{u}_f$ (see Eq. (2.21) of Ref. [3]), where $-e$ is the electron charge. Including a slow contribution to the magnetic field could be an important improvement, but this is outside the scope of the present work.

The validity of the above assumptions should be discussed in more depth. In the spirit of the fluid approximation used to derive the Zakharov equations, both for the classical and quantum cases, kinetic effects are neglected. Therefore, a necessary condition is that any characteristic length should be greater than the Debye length, a delicate assumption here since quantum effects tend to be more effective at nanoscales. In addition, the existence of a slow time scale must be checked in each particular example. For instance, in principle the temporal scale of a Langmuir envelope can become comparable to the inverse of the electron plasma frequency, especially in collapsing scenarios. Also, the nonrelativistic assumption should be verified. Finally, the quantum hydrodynamic model equations do not have dissipation terms, which amounts to a nonstrongly coupling assumption. For a degenerate plasma, appropriately described by Fermi-type equation of state (1), due to Pauli blocking the collisionless regime is better achieved for high-density environments. As discussed in Ref. [29], to avoid the need to include collisional effects one should have a small quantum coupling parameter $g_Q = 2m_e e^2/[(3\pi^2)^{2/3}\hbar^2\varepsilon_0 n_0^{1/3}]$, where $\hbar$ is the scaled Planck constant and ε_0 is the vacuum permittivity. For $g_Q < 1$, necessarily $n_0 > 10^{32}$ m^{-3}, a condition which can be attained, for instance, in the next generation laser-plasma compression experiments [30]. Some other dense plasma existing in environments such as white dwarfs or neutron stars must be considered with reserve because of relativistic effects. Also, the core of massive planets is not sufficiently dense, to avoid strong coupling. Nevertheless, the results from the present model can have at least a qualitative agreement in these extra systems.

Following the usual approximations [3,4], the quantum corrected 3D Zakharov equations then read

$$2i\omega_{pe}\frac{\partial\tilde{\mathbf{E}}}{\partial t} - c^2\nabla \times (\nabla \times \tilde{\mathbf{E}}) + v_{Fe}^2\nabla(\nabla\cdot\tilde{\mathbf{E}}) = \frac{\delta n_s}{n_0}\omega_{pe}^2\tilde{\mathbf{E}} + \frac{\hbar^2}{4m_e^2}\nabla[\nabla^2(\nabla\cdot\tilde{\mathbf{E}})], \tag{2}$$

$$\frac{\partial^2\delta n_s}{\partial t^2} - c_s^2\nabla^2\delta n_s - \frac{\varepsilon_0}{4m_i}\nabla^2(|\tilde{\mathbf{E}}|^2) + \frac{\hbar^2}{4m_e m_i}\nabla^4\delta n_s = 0. \tag{3}$$

Here $\tilde{\mathbf{E}}$ is the slowly varying envelope electric field defined via

$$\mathbf{E}_f = \frac{1}{2}(\tilde{\mathbf{E}}e^{-i\omega_{pe}t} + \tilde{\mathbf{E}}^* e^{i\omega_{pe}t}), \tag{4}$$

where ω_{pe} is the electron plasma frequency. Also, in Eqs. (2) and (3) c is the speed of light in vacuum and m_i is the ion mass. In addition, $c_s^2 = \kappa_B T_{Fe}/m_i$, where $\kappa_B T_{Fe} = m_e v_{Fe}^2$. Therefore, c_s is a Fermi ion-acoustic speed, with the Fermi temperature replacing the thermal temperature for the electrons.

In comparison to the classical Zakharov system (see Eqs. (2.48a) and (2.48b) of Ref. [3]), there is the inclusion of the extra dispersive terms proportional to $\hbar^2$ in Eqs. (2) and (3). Other quantum difference is the presence of the Fermi speed instead of the thermal speed in the last term at the left-hand side of Eq. (2). From the qualitative point of view, the terms proportional to $\hbar^2$ are responsible for extra dispersion which can avoid collapsing of Langmuir envelopes, at least in principle. This possibility is investigated in Secs. IV and V. Finally, notice the nontrivial form of the fourth-order derivative term in Eq. (2). It is not simply proportional to $\nabla^4\tilde{\mathbf{E}}$ as could be wrongly guessed from the quantum Zakharov equations in 1+1 dimensions, where there is a $\sim\partial^4\tilde{\mathbf{E}}/\partial x^4$ contribution [4].

It is useful to consider the rescaling

$$\bar{\mathbf{r}} = \frac{2\sqrt{\mu}\omega_{pe}\mathbf{r}}{v_{Fe}}, \quad \bar{t} = 2\mu\omega_{pe}t,$$

$$n = \frac{\delta n_s}{4\mu n_0}, \quad \mathcal{E} = \frac{e\tilde{\mathbf{E}}}{4\sqrt{\mu}m_e\omega_{pe}v_{Fe}}, \tag{5}$$

where $\mu = m_e/m_i$. Then, dropping the bars in $\mathbf{r}, t$, we obtain

$$i\frac{\partial\mathcal{E}}{\partial t} - \frac{c^2}{v_{Fe}^2}\nabla \times (\nabla \times \mathcal{E}) + \nabla(\nabla\cdot\mathcal{E}) = n\mathcal{E} + \Gamma\nabla[\nabla^2(\nabla\cdot\mathcal{E})], \tag{6}$$

$$\frac{\partial^2 n}{\partial t^2} - \nabla^2 n - \nabla^2(|\mathcal{E}|^2) + \Gamma\nabla^4 n = 0, \tag{7}$$

where

$$\Gamma = \frac{m_e}{m_i}\left(\frac{\hbar\omega_{pe}}{\kappa_B T_{Fe}}\right)^2 \tag{8}$$

is a nondimensional parameter associated with the quantum effects. Usually, it is an extremely small quantity, but it is nevertheless interesting to retain the $\sim\Gamma$ terms, especially for the collapse scenarios. The reason is not only due to a general theoretical motivation, but also because from some simple estimates one concludes that these terms become of the same order as some of other terms in Eqs. (2) and (3) provided that the characteristic length of the spatial derivatives becomes as small as the mean interparticle distance. To reach this conclusion, the expressions $\omega_{pe}=(n_0 e^2/m_e\varepsilon_0)^{1/2}$ and $\kappa_B T_{Fe}=\hbar^2(3\pi^2 n_0)^{2/3}/2m_e$ for the electron plasma frequency and Fermi energy were used. Of course, the Zakharov equations are not able to describe the late stages of the collapse since they do not include dissipation, which is unavoidable for short scales. However, even Landau damping would be irrelevant for a zero-temperature Fermi plasma, where the main influence comes from the Pauli pressure. Finally, in the left-hand side of Eq. (6) the $\nabla(\nabla\cdot\mathcal{E})$ term is retained because the $\sim c^2/v_{Fe}^2$ transverse term disappears in the electrostatic approximation.

For an hydrogen plasma, one has $\Gamma=5.64\times10^6 n_0^{-1/3}$, which gives $\Gamma\simeq10^{-4}$ for high densities such as $n_0=10^{32}\ \text{m}^{-3}$. However, this small numerical value of Γ does not justify to ignore the quantum effects. Rather, it is just a result from the scaling. Indeed, in collapsing scenarios the quantum terms play a decisive influence and have a magnitude comparable to the other terms in Eqs. (6) and (7). This property will be explicitly verified with simulations, in Sec. IV.

In the adiabatic limit, neglecting $\partial^2 n/\partial t^2$ in Eq. (7) and under appropriated boundary conditions, it follows that

$$n = -|\mathcal{E}|^2 + \Gamma\nabla^2 n. \tag{9}$$

When $\Gamma\neq0$, it is not immediate to directly express n as a function of $|\mathcal{E}|$ as in the classical case. Therefore, the adiabatic limit is not enough to derive a vector nonlinear Schrödinger equation, due to the coupling in Eq. (9).

The adiabatic (subsonic) assumption can be formulated as follows. Denoting by l the characteristic length and by τ the characteristic time scale for a nontranslating Langmuir wave packet, the issue is whether $l^2/\tau^2\ll1$, allowing discarding the second time-derivative term in Eq. (7). The inequality tends to be violated in collapsing scenarios, which (classically) are supersonic or self-similar. However, as shown later, quantum effects provide a qualitative change in the dynamics, avoiding collapse and, in this sense, extending the range of applicability of the subsonic regime. It is also of interest to verify if the time scale of the oscillations of the Langmuir wave packets is indeed slow enough in comparison to the plasma oscillations scale, so that $\omega_{pe}\tau\gg1$. Both necessary conditions must be checked in each concrete case.

III. LAGRANGIAN STRUCTURE AND CONSERVATION LAWS

The quantum Zakharov Eqs. (6) and (7) can be described by the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\left(\mathcal{E}^*\cdot\frac{\partial\mathcal{E}}{\partial t} - \mathcal{E}\cdot\frac{\partial\mathcal{E}^*}{\partial t}\right) - \frac{c^2}{v_{Fe}^2}|\nabla\times\mathcal{E}|^2 - |\nabla\cdot\mathcal{E}|^2 - \Gamma|\nabla(\nabla\cdot\mathcal{E})|^2 + n\left(\frac{\partial\alpha}{\partial t} - |\mathcal{E}|^2\right) - \frac{1}{2}(n^2 + \Gamma|\nabla n|^2 + |\nabla\alpha|^2), \tag{10}$$

where n, the auxiliary function α, and the components of $\mathcal{E},\mathcal{E}^*$ are regarded as independent fields. Remark: for the particular form [Eq. (10)] and for a generic field ψ, one computes the functional derivative as

$$\frac{\delta\mathcal{L}}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial r_i}\frac{\partial\mathcal{L}}{\partial\psi/\partial r_i} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\psi/\partial t} + \frac{\partial^2}{\partial r_i\,\partial r_j}\frac{\partial\mathcal{L}}{\partial^2\psi/\partial r_i\,\partial r_j}, \tag{11}$$

using the summation convention and where r_i are Cartesian components.

Taking the functional derivatives with respect to n and α, we have

$$\frac{\partial\alpha}{\partial t} = n + |\mathcal{E}|^2 - \Gamma\nabla^2 n \tag{12}$$

and

$$\frac{\partial n}{\partial t} = \nabla^2\alpha, \tag{13}$$

respectively. Eliminating α from Eqs. (12) and (13) we obtain the low-frequency equation. In addition, the functional derivatives with respect to $\mathcal{E}^*$ and $\mathcal{E}$ produce the high-frequency equation and its complex conjugate. The present formalism is inspired by the Lagrangian formulation of the classical Zakharov equations [31].

The quantum Zakharov equations admit as exact conserved quantities the "number of plasmons" of the Langmuir field,

$$N = \int |\mathcal{E}|^2 d\mathbf{r}, \tag{14}$$

the linear momentum (with components $P_i, i=x,y,z$),

$$P_i = \int\left[\frac{i}{2}\left(\mathcal{E}_j\frac{\partial\mathcal{E}_j^*}{\partial r_i} - \mathcal{E}_j^*\frac{\partial\mathcal{E}_j}{\partial r_i}\right) - n\frac{\partial\alpha}{\partial r_i}\right]d\mathbf{r}, \tag{15}$$

and the Hamiltonian,

$$\mathcal{H} = \int\left[n|\mathcal{E}|^2 + \frac{c^2}{v_{Fe}^2}|\nabla\times\mathcal{E}|^2 + |\nabla\cdot\mathcal{E}|^2 + \Gamma|\nabla(\nabla\cdot\mathcal{E})|^2 + \frac{1}{2}(n^2 + \Gamma|\nabla n|^2 + |\nabla\alpha|^2)\right]d\mathbf{r}. \tag{16}$$

Furthermore, there is also a preserved angular momenta functional, but it is not relevant in the present work. These four conserved quantities can be associated, through Noether's theorem, to the invariance of the action under gauge transformation, time translation, space translation, and rotations, respectively. The conservation laws can be used, e.g., to test the accuracy of numerical procedures. Also, observe that Eqs. (7) and (9) for the adiabatic limit are described by the same Lagrangian density [Eq. (10)]. In this approximation, it suffices to set $\alpha \equiv 0$.

In addition to the adiabatic limit, Eq. (9) can be further approximated to

$$n = -|\mathcal{E}|^2 - \Gamma \nabla^2(|\mathcal{E}|^2), \quad (17)$$

assuming that the quantum term is a perturbation. In this way and using Eq. (6), a quantum modified vector nonlinear Schrödinger equation is derived,

$$i\frac{\partial \mathcal{E}}{\partial t} + \nabla(\nabla \cdot \mathcal{E}) - \frac{c^2}{v_{Fe}^2} \nabla \times (\nabla \times \mathcal{E}) + |\mathcal{E}|^2 \mathcal{E} = \Gamma \nabla [\nabla^2(\nabla \cdot \mathcal{E})] - \Gamma \mathcal{E} \nabla^2(|\mathcal{E}|^2). \quad (18)$$

The appropriate Lagrangian density $\mathcal{L}_{ad,sc}$ for the semiclassical Eq. (18) is given by

$$\mathcal{L}_{ad,sc} = \frac{i}{2}\left(\mathcal{E}^* \cdot \frac{\partial \mathcal{E}}{\partial t} - \mathcal{E} \cdot \frac{\partial \mathcal{E}^*}{\partial t}\right) - \frac{c^2}{v_{Fe}^2}|\nabla \times \mathcal{E}|^2 - |\nabla \cdot \mathcal{E}|^2 - \Gamma|\nabla(\nabla \cdot \mathcal{E})|^2 + \frac{1}{2}|\mathcal{E}|^4 - \frac{\Gamma}{2}|\nabla[|\mathcal{E}|^2]|^2, \quad (19)$$

where the independent fields are taken as the $\mathcal{E}$ and $\mathcal{E}^*$ components.

The expression N for the number of plasmons in Eq. (14) remains valid as a constant of motion in the joint adiabatic and semiclassical limit, as well as the momentum $\mathbf{P}$ in Eq. (15) with $\alpha \equiv 0$. Finally, the Hamiltonian

$$\mathcal{H}_{ad,sc} = \int \left[\frac{c^2}{v_{Fe}^2}|\nabla \times \mathcal{E}|^2 + |\nabla \cdot \mathcal{E}|^2 + \Gamma|\nabla(\nabla \cdot \mathcal{E})|^2 - \frac{1}{2}|\mathcal{E}|^4 + \frac{\Gamma}{2}|\nabla[|\mathcal{E}|^2]|^2\right] d\mathbf{r} \quad (20)$$

is also a conserved quantity.

In terms of the scale length l of the Langmuir envelopes, it follows from both Eqs. (6) and (7) that the quantum terms are higher-order corrections provided $\Gamma \ll l^2$. In view of the smallness of the parameter Γ in high-density plasmas, this is a not so stringent inequality, but can imply limitations especially in quasicollapse scenarios. Nevertheless, the quantum terms imply qualitative changes in the dynamics of self-focusing Langmuir wave packets, as explained in the next sections.

In the following, the influence of the quantum terms in the right-hand side of Eq. (18) is investigated, assuming adiabatic and semiclassical conditions for collapsing quantum Langmuir envelopes. Other collapse regimes, such as the supersonic one [18,19], are also relevant and shall be investigated in the future. The present approximations are mainly justified on a relative analytical simplicity. Certainly, more detailed advances can be obtained via numerical simulation of the full quantum Zakharov system. However, direct numerical integration of the equations is made delicate by the presence of the high-order dispersive terms that plays a central role on the small-scale dynamics.

IV. VARIATIONAL SOLUTION IN TWO DIMENSIONS

Consider the adiabatic semiclassical system defined by Eq. (18). We refer to localized solution for this vector NLS equation as (quantum) "Langmuir wave packets," or envelopes. As discussed in detail in [31] in the purely classical case, Langmuir wave packets will become singular in a finite time, provided the energy is not bounded from below. Of course, explicit analytic Langmuir envelopes are difficult to derive. A fruitful approach is to make use of the Lagrangian structure for deriving approximate solutions. This approach has been pursued in [32] for the classical and in [9] for the quantum Zakharov system. Both studies considered the internal vibrations of Langmuir envelopes in one spatial dimension. Presently, we shall apply the time-dependent Rayleigh-Ritz method for the higher-dimensional cases. *A priori*, it is expected that the quantum corrections would inhibit the collapse of localized solutions, in view of wave-packet spreading. To check this conjecture, and to have more definite information on the influence of the quantum terms, first we consider the following *Ansatz*,

$$\mathcal{E} = \left(\frac{N}{\pi}\right)^{1/2} \frac{1}{\sigma} \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \exp[i(\Theta + k\rho^2)](\cos\phi, \sin\phi, 0), \quad (21)$$

which is appropriate for two spatial dimensions. Here σ, k, Θ, and ϕ are real functions of time, and $\rho = \sqrt{x^2 + y^2}$. Normalization condition (14) is automatically satisfied (in two dimensions the spatial integrations reduce to integrations on the plane). Other localized forms, involving, e.g., a sech-type dependence, could have been also proposed. Here a Gaussian form was suggested mainly for the sake of simplicity [33]. Notice that envelope electric field (21) is not necessarily electrostatic: it can carry a transverse ($\nabla \times \mathcal{E} \neq 0$) component.

The free functions in Eq. (21) should be determined by extremization of the action functional associated with Lagrangian density (19). A straightforward calculation gives

$$L_2 \equiv \int \mathcal{L}_{ad,sc} dx dy = -N\left[\dot{\Theta} + \sigma^2 \dot{k} + \frac{2c^2}{v_{Fe}^2} k^2 \sigma^2 + \frac{1}{2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^2} + 8\Gamma k^2 + 16\Gamma k^4 \sigma^4 + \left(1 + \frac{N}{2\pi}\right)\frac{\Gamma}{\sigma^4}\right], \quad (22)$$

where only the main quantum contributions are retained. Now L_2 is the Lagrangian for a mechanical system, after the spatial form of the envelope electric field was defined in advance via Eq. (21). Of special interest is the behavior of the dispersion σ. For a collapsing solution one could expect that σ goes to zero in a finite time. The phase Θ and the chirp function k should be regarded as auxiliary fields. Notice that L_2 is not dependent on the angle ϕ, which remains arbitrary as far as the variational method is concerned.

Applying the functional derivative of L_2 with respect to Θ, we obtain

$$\frac{\delta L_2}{\delta \Theta} = 0 \rightarrow \dot{N} = 0, \tag{23}$$

so that the variational solution preserves the number of plasmons, as expected. The remaining Euler-Lagrange equations are

$$\frac{\delta L_2}{\delta k} = 0 \rightarrow \sigma\dot{\sigma} = \frac{2c^2}{v_{Fe}^2}\sigma^2 k + 8\Gamma k + 32\Gamma\sigma^4 k^3, \tag{24}$$

$$\frac{\delta L_2}{\delta \sigma} = 0 \rightarrow \sigma\dot{k} = -\frac{2c^2}{v_{Fe}^2}k^2\sigma + \frac{1}{2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^3} - 32\Gamma k^4\sigma^3 + \left(1 + \frac{N}{2\pi}\right)\frac{2\Gamma}{\sigma^5}. \tag{25}$$

The exact solution of the nonlinear system [Eqs. (24) and (25)] is difficult to obtain, but at least the dynamics was reduced to ordinary differential equations.

It is instructive to analyze the purely classical ($\Gamma \equiv 0$) case first. This is especially important since to our knowledge the Rayleigh-Ritz method was not applied to vector NLS Eq. (18), even in the classical limit. The reason can be due to the calculational complexity induced by the transverse term. When $\Gamma=0$, Eq. (24) gives $k = v_{Fe}^2\dot{\sigma}/2c^2\sigma$. Inserting this in Eq. (25) we have

$$\ddot{\sigma} = -\frac{\partial V_{2c}}{\partial \sigma}, \tag{26}$$

where the pseudopotential V_{2c} is

$$V_{2c} = \frac{c^2}{2v_{Fe}^2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^2}. \tag{27}$$

From Eq. (27) it is evident that the repulsive character of the pseudopotential will be converted into an attractive one, whenever the number of plasmons exceeds a threshold,

$$N > \frac{2\pi c^2}{v_{Fe}^2}, \tag{28}$$

a condition for Langmuir wave-packet collapse in the classical two-dimensional case. The interpretation of the result is as follows. When the number of plasmons satisfy Eq. (28), the refractive $\sim|\mathcal{E}|^4$ term dominates over the dispersive terms in Lagrangian density (19), producing a singularity in a finite time. In addition, notice that in the case of a classical ideal gas equation of state, the only basic change would be the replacement of the Fermi by the thermal velocity. Finally, notice the ballistic motion when $N = 2\pi c^2/v_{Fe}^2$, which can also lead to singularity.

Further insight follows after evaluating energy integral (20) with *Ansatz* (21), which gives, after eliminating k,

$$\mathcal{H}_{ad,sc,2c} = \frac{Nv_{Fe}^2}{c^2}\left[\frac{\dot{\sigma}^2}{2} + V_{2c}\right] \quad (\Gamma \equiv 0). \tag{29}$$

Of course, this energy first integral could be obtained directly from Eq. (26). However, the plausibility of the variational solution is reinforced since Eq. (29) shows that it preserves the exact constant of motion $\mathcal{H}_{ad,sc}$. In addition, in the attractive (collapsing) case energy (29) is not bounded from bellow.

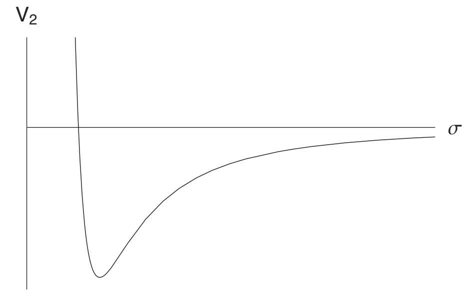

FIG. 1. The qualitative form of the pseudopotential in Eq. (31) for $N > 2\pi c^2/v_{Fe}^2$.

In the quantum ($\Gamma \neq 0$) case, Eq. (24) becomes a cubic equation in k, whose exact solution is too cumbersome to be of practical use. It is better to proceed by successive approximations, taking into account that the quantum and electromagnetic terms are small. In this way, one arrives at

$$\ddot{\sigma} = -\frac{\partial V_2}{\partial \sigma}, \tag{30}$$

where the pseudopotential V_2 is

$$V_2 = \frac{c^2}{2v_{Fe}^2}\left(\frac{c^2}{v_{Fe}^2} - \frac{N}{2\pi}\right)\frac{1}{\sigma^2} + \frac{\Gamma c^2}{v_{Fe}^2}\left(1 + \frac{N}{2\pi}\right)\frac{1}{\sigma^4}. \tag{31}$$

Now, even if threshold (28) is exceeded, the repulsive $\sim\sigma^{-4}$ quantum term in V_2 will prevent singularities. This adds quantum diffraction as another physical mechanism, besides dissipation and Landau damping so that collapsing Langmuir wave packets are avoided in the vector NLS equation. Also, similarly to Eq. (29), it can be shown that the approximate dynamics preserves the energy integral, even in the quantum case. Indeed, calculating from Eq. (20) and the variational solution gives $\mathcal{H}_{ad,sc}$ as

$$\mathcal{H}_{ad,sc,2} = \frac{Nv_{Fe}^2}{c^2}\left[\frac{\dot{\sigma}^2}{2} + V_2\right] \quad (\Gamma \geq 0). \tag{32}$$

From Eq. (30), obviously $\dot{\mathcal{H}}_{ad,sc,2} = 0$.

It should be noticed that oscillations of purely quantum nature are obtained when the number of plasmons exceeds threshold (28). Indeed, in this case the pseudopotential V_2 in Eq. (31) assumes a potential well form as shown in Fig. 1, which clearly admits oscillations around a minimum $\sigma = \sigma_m$. Here,

$$\sigma_m = 2\left[\frac{\Gamma(1 + N/2\pi)}{N/2\pi - c^2/v_{Fe}^2}\right]^{1/2}. \tag{33}$$

Also, the minimum value of V_2 is

$$V_2(\sigma_m) = -\frac{c^2}{16\Gamma v_{Fe}^2}\frac{(N/2\pi - c^2/v_{Fe}^2)^2}{1 + N/2\pi} > -\frac{1}{16\Gamma}\left(\frac{N}{2\pi} - \frac{c^2}{v_{Fe}^2}\right)^2, \tag{34}$$

the last inequality following since Eq. (28) is assumed. Therefore, a deepest potential well is obtained for increasing

N. Also, for too large quantum effects the trapping of the localized electric field in this potential well would be difficult since $V_2(\sigma_m)\to 0_-$ as Γ increases. This is due to the dispersive nature of the quantum corrections.

The frequency Ω of the small amplitude oscillations is derived linearizing Eq. (30) around equilibrium point (33), yielding

$$\Omega = \frac{c}{2\sqrt{2}\Gamma v_{Fe}}\frac{(N/2\pi - c^2/v_{Fe}^2)^{3/2}}{1+N/2\pi}. \tag{35}$$

Restoring physical coordinates via Eq. (5) this frequency is calculated as

$$\omega = \frac{c}{\sqrt{2}v_{Fe}}\left(\frac{\kappa_B T_{Fe}}{\hbar\omega_{pe}}\right)^2\frac{(N/2\pi - c^2/v_{Fe}^2)^{3/2}}{1+N/2\pi}\omega_{pe} < \frac{v_{Fe}}{\sqrt{2}c}\left(\frac{\kappa_B T_{Fe}}{\hbar\omega_{pe}}\right)^2\left(\frac{N}{2\pi}-\frac{c^2}{v_{Fe}^2}\right)^{3/2}\omega_{pe}. \tag{36}$$

Equation (36) gives the frequency of the breatherlike mode of quantum Langmuir wave packets. To be coherent with the slow temporal dynamics of the envelope electric field and the density fluctuation, we impose $\omega/\omega_{pe}\ll 1$. Examining Eq. (36), we find a smaller oscillation frequency for bigger ambient density n_0 and for smaller electrostatic energy (which is proportional to N). Hence, to fit the model it is better to not have an extremely high plasmon number N, otherwise one would not be in the scope of the slow time-dependence approximation. However, the plasmon number cannot be too small because inequality (28) is supposed to hold, for the existence of the breather mode. In this spirit, it is convenient to define a parameter $0<\delta<1$ such that

$$\frac{N}{2\pi}\equiv(1+\delta)\frac{c^2}{v_{Fe}^2}. \tag{37}$$

Combining Eqs. (36) and (37), one has, to good agreement and using the nonrelativistic condition,

$$\frac{\omega}{\omega_{pe}} = \frac{10^{14}}{1+\delta}\frac{\delta^{3/2}}{n_0^{1/3}}, \tag{38}$$

for an hydrogen plasma. One can verify that for ordinary plasmas the oscillation frequency of the breather mode is too high. Hence, we assume a dense plasma with $n_0=2\times 10^{33}$ m^{-3}, which is expected to be realized in the next generation intense laser-solid plasma compression schemes. In this case, from Eq. (38) one needs $\delta<10^{-4}$ to comply with $\omega/\omega_{pe}\ll 1$. Specifically, we define $\delta=0.5\times 10^{-4}$ and, additionally, a temperature $T=10^7$ K, a realistic value in resonant backward Raman amplification of laser pulses in plasmas [30]. In this case, one has the parameters $\omega_{pe}=2.52\times 10^{18}$ s^{-1} and $v_{Fe}=0.11c$, three times the electron thermal speed and still weakly relativistic. It is essential to have a small thermal velocity since equation of state (1) for a zero-temperature Fermi gas was used as starting point. Moreover, $\omega/\omega_{pe}=2.65\times 10^{-4}$, fairly small to be in accordance with the slow time-dependence hypothesis. Finally, one found an electrostatic energy $(\varepsilon_0/2)\int|\mathcal{E}|^2 d\mathbf{r}=0.57$ GeV and a quantum parameter $\Gamma=4.48\times 10^{-5}$. Such a small number should not be overemphasized since the nonpurely classical dynamics is quite different from the strictly classical one.

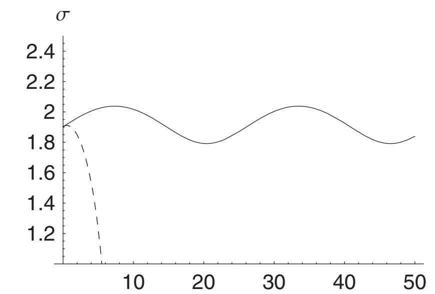

FIG. 2. Numerical solution for the width σ satisfying Eqs. (24) and (25). Parameters: $n_0=2\times 10^{33}$ m^{-3} and $(\varepsilon_0/2)\int|\mathcal{E}|^2 d\mathbf{r}=0.57$ GeV. Initial conditions: $\sigma(0)=1.90$ and $k(0)=0.05$. Dashed curve: the corresponding collapsing classical solution without the quantum terms.

The adiabatic condition should be also verified. Due to the form of the trial function in Eq. (21) and $|n|\sim\mathcal{E}^2$ to first approximation, one can assign the spatial scale $l\sim\sigma$ to the density fluctuation n. Moreover, in the case of oscillatory wave packets, which are the more interesting since they are a quantum-mechanical effect, one has the temporal scale $\tau\sim\Omega^{-1}$, where Ω is given by Eq. (35). In addition, for not too strongly nonlinear situations, one can use $\sigma\sim\sigma_m$, where σ_m is given by Eq. (33). With the chosen parameters, we have $l^2/\tau^2\sim\Omega^2\sigma_m^2\sim 0.22$, in satisfactory agreement with a subsonic dynamics. However, for nonlinear oscillations far from the equilibrium point, both estimates $\tau\sim\Omega^{-1}$ and $\sigma\sim\sigma_m$ can be questioned and tested numerically. It could be tempting to take a bigger δ, which of course would be easier to achieve in experiments, but then the adiabatic approximation would not be satisfied. This is because the oscillation frequency of the breather mode becomes bigger for bigger plasmon number. In this respect this work is just a first attempt to characterize the periodic behavior of quantum Langmuir wave packets. Figure 2 shows the behavior of the width σ solving Eqs. (24) and (25), with initial condition near the fixed point $(\sigma_m,k)=(1.90,0)$. For comparison, also the collapsing classical trajectory is shown, obtained removing the quantum terms proportional to Γ in the same system. Figure 3 shows the corresponding periodic trajectory in phase space (σ,k).

The remaining conditions to be checked are the smallness of the quantum terms and the neglect of kinetic and strong-coupling effects. For the assumed high density $n_0=2$

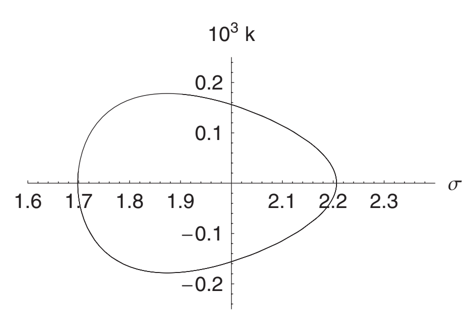

FIG. 3. Numerical solution for Eqs. (24) and (25) in phase space, with the same parameters as in Fig. 2.

$\times 10^{33}$ m^{-3}, strong-coupling effects can be safely ignored in view of Pauli blocking which avoid collisions, as argued in Sec. I. Moreover, using rescaling (5), one gets a nondimensional Debye length $\lambda_D = 0.02$, much smaller than the characteristic length $l \sim \sigma_m = 1.90$, so that the fluid approximation is satisfactory. Finally, from both Eqs. (6) and (7) we conclude that the quantum terms are perturbations provided $l^2 \gg \Gamma$, which presently traduces into $3.61 \gg 4.48 \times 10^{-5}$.

To conclude the section, the variational solution suggests that the extra dispersion arising from the quantum terms would inhibit the collapse of Langmuir wave packets in two spatial dimensions. Moreover, for sufficient electric field energy (which is proportional to N), instead of collapse there will be oscillations of the width of the localized solution, due to the competition between the classical refraction and the quantum diffraction. The frequency of linear oscillations is then given by Eq. (36). The emergence of a pulsating Langmuir envelope could be tested quantitatively in experiments, such as those involving the next generation intense laser-solid density plasma compression schemes.

V. VARIATIONAL SOLUTION IN THREE DIMENSIONS

It is worth to study the dynamics of localized solutions for vector NLS Eq. (18) in fully three-dimensional space. For this purpose, we consider the Gaussian form

$$\mathcal{E} = \left(\frac{N}{(\sqrt{\pi}\sigma)^3}\right)^{1/2} \exp\left[-\frac{r^2}{2\sigma^2} + i(\Theta + kr^2)\right] \times (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \qquad (39)$$

where σ, k, Θ, θ, and ϕ are real functions of time and $r = \sqrt{x^2+y^2+z^2}$, applying the Rayleigh-Ritz method just like in the last section. Normalization condition (14) is automatically satisfied with Eq. (39), which, occasionally, can also support a transverse ($\nabla \times \mathcal{E} \neq 0$) part.

Proceeding as before, the Lagrangian

$$L_3 \equiv \int \mathcal{L}_{ad,sc} d\mathbf{r} = -N\left[\dot{\Theta} + \frac{3}{2}\sigma^2\dot{k} + \frac{4c^2}{v_{Fe}^2}k^2\sigma^2 + \frac{c^2}{v_{Fe}^2\sigma^2} - \frac{N}{4\sqrt{2}\pi^{3/2}\sigma^3} + 10\Gamma k^2 + 20\Gamma k^4\sigma^4 + \frac{5\Gamma}{4\sigma^4} + \frac{3\Gamma N}{4\sqrt{2}\pi^{3/2}\sigma^5}\right] \qquad (40)$$

is derived. In comparison to the reduced two-dimensional Lagrangian in Eq. (22), there are different numerical factors as well as qualitative changes due to higher-order nonlinearities. Also, the angular variables θ and ϕ do not appear in L_3.

The main remaining task is to analyze the dynamics of the width σ as a function of time. This is achieved from the Euler-Lagrange equations for the action functional associated to L_3. As before, $\delta L_3/\delta\Theta = 0$ gives $\dot{N} = 0$, a consistency test satisfied by the variational solution. The other functional derivatives yield

$$\frac{\delta L_3}{\delta k} = 0 \rightarrow \sigma\dot{\sigma} = \frac{4k}{3}\left[\frac{2c^2}{v_{Fe}^2}\sigma^2 + 5\Gamma(1 + 4k^2\sigma^4)\right], \qquad (41)$$

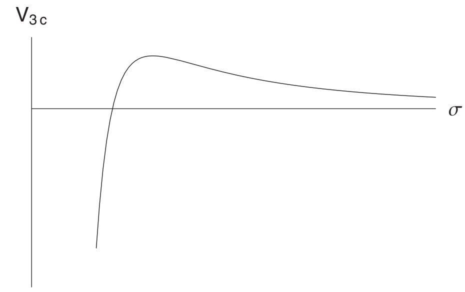

FIG. 4. The qualitative form of the pseudopotential V_{3c} in Eq. (44).

$$\frac{\delta L_3}{\delta \sigma} = 0 \rightarrow \sigma\dot{k} = \frac{1}{3}\left[-\frac{8c^2}{v_{Fe}^2}k^2\sigma + \frac{2c^2}{v_{Fe}^2\sigma^3} - \frac{3N}{4\sqrt{2}\pi^{3/2}\sigma^4} - 80\Gamma k^4\sigma^3 + \frac{5\Gamma}{\sigma^5} + \frac{15\Gamma N}{4\sqrt{2}\pi^{3/2}\sigma^6}\right]. \qquad (42)$$

In the formal classical limit ($\Gamma \equiv 0$), and using Eq. (41) to eliminate k, we obtain

$$\ddot{\sigma} = -\frac{\partial V_{3c}}{\partial \sigma}, \qquad (43)$$

where now the pseudopotential V_{3c} is

$$V_{3c} = \frac{c^2}{v_{Fe}^2}\left(\frac{8c^2}{9v_{Fe}^2\sigma^2} - \frac{2N}{9\sqrt{2}\pi^{3/2}\sigma^3}\right). \qquad (44)$$

Form (44) shows a generic singular behavior since the attractive $\sim\sigma^{-3}$ term will dominate for sufficiently small σ, irrespective of the value of N. Hence, in fully three-dimensional space there is more "room" for a collapsing dynamics. Figure 4 shows the qualitative form of V_{3c}, attaining a maximum at $\sigma = \sigma_M$, where

$$\sigma_M = \frac{3v_F^2 N}{8\sqrt{2}\pi^{3/2}c^2}. \qquad (45)$$

By Eq. (42) and using successive approximations in the parameter Γ to eliminate k via Eq. (41), we obtain

$$\ddot{\sigma} = -\frac{\partial V_3}{\partial \sigma}, \qquad (46)$$

where

$$V_3 = \frac{8c^2}{3v_{Fe}^2}\left[\frac{c^2}{3v_{Fe}^2\sigma^2} - \frac{N}{12\sqrt{2}\pi^{3/2}\sigma^3} + \frac{5\Gamma}{12\sigma^4} + \frac{\Gamma N}{4\sqrt{2}\pi^{3/2}\sigma^5}\right]. \qquad (47)$$

The quantum terms are repulsive and prevent collapse since they dominate for sufficiently small σ. Moreover, when $\Gamma \neq 0$ an oscillatory behavior is possible, provided a certain condition, to be explained in the following, is meet.

To examine the possibility of oscillations, consider $V_3'(\sigma) = 0$, the equation for the critical points of V_3. Under the rescaling $s = \sigma/\sigma_M$, where σ_M [defined in Eq. (45)] is the maximum of the purely classical pseudopotential, the equation for the critical points reads

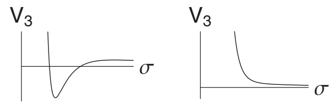

FIG. 5. The qualitative form of the pseudopotential V_3 in Eq. (47) for $g<1$ (on the left) and $g>1$ (on the right).

$$V_3' = 0 \to s^3 - s^2 + \frac{4g}{27} = 0, \tag{48}$$

where

$$g = \frac{480\pi^3 \Gamma c^4}{N^2 v_{Fe}^4} \tag{49}$$

is a new dimensionless parameter. In deriving Eq. (48), it was omitted a term negligible except if $s \sim c^2/v_{Fe}^2$, which is unlikely.

The quantity g plays a decisive role on the shape of V_3. Indeed, calculating the discriminant shows that the solutions to the cubic in Eq. (48) are as follows: (a) $g<1 \to$ three distinct real roots (one negative and two positive); (b) $g=1$ $\to$ one negative root, one (positive) double root; (c) $g>1$ $\to$ one (negative) real root, two complex-conjugate roots. Therefore, $g<1$ is the condition for the existence of a potential well, which can support oscillations. This is shown in Fig. 5. The analytic formulae for the solutions of the cubic in Eq. (48) are cumbersome and will be omitted.

Restoring physical coordinates, the necessary condition for oscillations is rewritten as

$$g < 1 \to \frac{\varepsilon_0}{2} \int |\tilde{\mathbf{E}}|^2 d\mathbf{r} > \frac{\sqrt{30\pi}}{\gamma} m_e v_{Fe} c, \tag{50}$$

where $\gamma = e^2/4\pi\varepsilon_0\hbar c \simeq 1/137$ is the fine-structure constant. From Eq. (50) it is seen that for sufficient electrostatic energy the width σ of the localized envelope field can show oscillations, supported by the competition between classical refraction and quantum diffraction. Also, due to the Fermi pressure, for large particle densities inequality (50) becomes more difficult to be met since $v_{Fe} \sim n_0^{1/3}$. For example, when $n_0 \sim 10^{36}$ m^{-3} (white dwarf), the right-hand-side of Eq. (50) is 0.6 GeV. For $n_0 \sim 10^{33}$ m^{-3} (the next generation intense laser-solid density plasma experiments), it is 57.5 MeV.

Finally, notice that $\mathcal{H}_{ad,sc}$ from Eq. (20), evaluated with variational solution (39), is proportional to $\dot{\sigma}^2/2 + V_3$, which is a constant of motion for Eq. (46). Therefore, the approximate solution preserves one of the basic first integrals of vector NLS Eq. (18), as it should do.

VI. CONCLUSION

In this paper, the quantum Zakharov system in fully three-dimensional space has been derived. An associated Lagrangian structure was found, as well as the pertinent conservation laws. From the Lagrangian formalism, many possibilities are opened. Here, the variational description was used to analyze the behavior of localized envelope electric fields of Gaussian shape, in both two- and three-space dimensions. It was shown that the quantum corrections induce qualitative and quantitative changes, inhibiting singularities and allowing for oscillations of the width of the Langmuir envelope field. This dynamics can be tested in experiments with focusing of intense x-ray pulses by means of resonant backward Raman amplification in plasmas. Indeed, in the next generation laser-based plasma compression schemes, large particle densities such as $n_e \sim 10^{33}$ m^{-3} can be achievable (there is no known theoretical limit [30]). In particular, the influence of the parameter g and the inequality in Eq. (50) should be investigated. However, the variational method was applied here only for the adiabatic and semiclassical case, which allows deriving quantum modified vector NLS Eq. (18). Other more general scenarios for the solutions of the fully three-dimensional quantum Zakharov system are also worth to study, with numerical and real experiments. In this regard, it could be very interesting to apply the Lagrangian method to the full quantum Zakharov equations, without limitation to the adiabatic or semiclassical case. For this it would be necessary to specify not only the envelope electric field trial function, but also the extra fields n and α present in the Lagrangian density in Eq. (10). The appropriate choice of these additional trial functions is not completely trivial and will be postponed.

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